

# GREEDY STRATEGIES FOR CONVEX OPTIMIZATION

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**ABSTRACT.** We investigate two greedy strategies for finding an approximation to the minimum of a convex function  $E$  defined on a Hilbert space  $H$ . We prove convergence rates for these algorithms under suitable conditions on the objective function  $E$ . These conditions involve the behavior of the modulus of smoothness and the modulus of uniform convexity of  $E$ .

**Key Words:** Greedy Algorithms, Convex Optimization, Rates of Convergence.

## 1. INTRODUCTION

Convex optimization has many application domains such as automatic control systems, signal processing, communications and networks, electronic circuit design, data analysis and modeling, statistical estimation, finance, and combinatorial optimization. A general description for convex optimization is that we are given a Banach space  $X$  and a convex function  $E$  on  $X$  whose minimum we wish to compute. Thus, we are interested in the development and analysis of algorithms for approximating

$$(1.1) \quad \inf_{x \in D} E(x),$$

where  $D$  is a convex subset of  $X$ .  $E$  is called the *objective* function and, by the convexity assumption, satisfies the condition

$$E(\gamma x + \delta y) \leq \gamma E(x) + \delta E(y), \quad x, y \in D, \quad \gamma, \delta \geq 0, \quad \gamma + \delta = 1.$$

The classical results on convex optimization deal with objective functions  $E$  defined on subsets in  $\mathbb{R}^d$  with moderate values of  $d$ , see e.g. [2]. However, several of the applications, listed above, lead to optimization on Banach spaces of dimension  $d$ , where  $d$  is quite large or even  $\infty$ . The design of algorithms for such high dimensional problems is quite challenging, typical convergent results involve the dimension  $d$  and suffer from the curse of dimensionality.

Recently, several researchers (see e.g. [7, 8, 11]), have proposed strategies for solving (1.1), where the curse of dimensionality is overcome by using greedy techniques, similar to those originally developed for the approximation of a given element  $x \in X$ .

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The minimum in (1.1) is approximated by  $E(x_m)$ ,  $m = 0, 1, \dots$ , where each  $x_m$  is constructed as a linear combination of  $m$  elements (i.e.  $x_m$  is  $m$  *sparse*) from a given dictionary  $\mathcal{D}$ . Recall that  $\mathcal{D}$  is called a symmetric dictionary if each  $\varphi \in \mathcal{D}$  has norm  $\|\varphi\| \leq 1$ , if  $\varphi \in \mathcal{D}$ , then  $-\varphi \in \mathcal{D}$ , and the closure of  $\text{span } \mathcal{D}$  is  $X$ . A typical a priori convergence result given by the above authors for these greedy algorithms is proven under two assumptions:

- (i) An assumption on the smoothness of  $E$ .
- (ii) An assumption that the minimum in (1.1) is taken at a point  $\bar{x}$  which is in the convex hull of the dictionary  $\mathcal{D}$ .

In this paper, we investigate the special case when  $X = H$  is a Hilbert space, the dictionary  $\mathcal{D} = \{\pm\varphi_j\}_{j=1}^\infty$ , where  $\{\pm\varphi_j\}_{j=1}^\infty$  is an orthonormal basis, and  $D = H$  (which corresponds to global minimization). We assume that the global minimum is attained at some point  $\bar{x} \in H$ . It follows then that the minimum is taken on the set

$$\Omega := \{x \in H : E(x) \leq E(0)\}.$$

We assume throughout this paper that the set  $\Omega$  is bounded in  $H$ . We impose the following assumptions on the objective function  $E$ :

**Condition 0:**  $E$  has a Frechet derivative  $E'(x) \in H$  at each point  $x$  in  $\Omega$  and

$$\|E'(x)\| \leq M_0, \quad x \in \Omega,$$

where throughout  $\|\cdot\|$  denotes the norm on  $H$ .

**Condition 1:** There are constants  $0 < \alpha$ ,  $1 < q \leq 2$  and  $0 < M$ , such that for all  $x, x'$  with  $\|x - x'\| \leq M$ ,  $x \in \Omega$ ,

$$(1.2) \quad E(x') - E(x) - \langle E'(x), x' - x \rangle \leq \alpha \|x' - x\|^q.$$

**Condition 2:** There are constants  $0 < \beta$ ,  $2 \leq p < \infty$  and  $0 < M$ , such that for all  $x, x'$  with  $\|x - x'\| \leq M$ ,  $x \in \Omega$ ,

$$(1.3) \quad E(x') - E(x) - \langle E'(x), x' - x \rangle \geq \beta \|x' - x\|^p.$$

We show in §2 that **Condition 1** is equivalent to conditions on the modulus of smoothness  $\rho(E, u)$ , and **Condition 2** is equivalent to conditions on the modulus of uniform convexity  $\delta_1(E, u)$ , as usually defined in convex optimization (see e.g. [10]), and introduced by us in §2.

We study two greedy procedures for solving (1.1). The first is the analogue for convex minimization of the Orthogonal Matching Pursuit Algorithm used for approximation (see [6]). We denote this convex minimization algorithm by OMP(co)<sup>1</sup>.

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<sup>1</sup>Here and later we will use the abbreviation (co) if an algorithm is used for convex optimization

The second is the Weak Chebyshev Greedy Algorithm (WCGA(co)) as introduced by Temlyakov [8]. These greedy procedures, which are defined in §3, iteratively generate a sequence  $x_m$ ,  $m = 0, 1, \dots$ , where each  $x_m$  is  $m$  sparse, and then use  $E(x_m)$  as the approximation to the minimum  $E(\bar{x})$ .

Our main results are Theorem 4.5 and Theorem 4.6 which establish a priori convergence rates for both OMP(co) and the WCGA(co) when they are used to find the minimum of a function  $E$  that satisfies **Conditions 0, 1** and **2**. For example, we show that if the objective function  $E$  satisfies **Condition 0** and **Condition 1**, is strongly convex on  $H$  (therefore satisfies **Condition 2** with  $p = 2$ ), and its minimizer  $\bar{x}$  is sparse with respect to  $\mathcal{D}$ , then the error at the  $m$ -th step of the OMP(co) satisfies the inequality

$$E(x_m) - E(\bar{x}) \leq C_0 m^{1 - \frac{q}{2-q}}, \quad 1 < q < 2,$$

and

$$\|x_m - \bar{x}\| \leq C_1 m^{\frac{1}{2} - \frac{q}{2(2-q)}},$$

where  $C_0 = C_0(q, E)$  and  $C_1 = C_1(q, E)$ . We also prove exponential convergence in the case  $q = 2$ . In contrast, the results from [8] and [11] do not impose **Condition 2** and only give the rate  $1 - q$ . In summary, we show that imposing more conditions on the convexity of the objective function  $E$  (like **Condition 2**) results in provably improved convergence rates for both OMP(co) and WCGA(co).

## 2. CONDITIONS ON $E$

In this section, we discuss the compatibility of the conditions (**Condition 0**, **Condition 1** and **Condition 2**) imposed on the objective function  $E$  and their relation to the modulus of smoothness and modulus of uniform convexity of  $E$ . We recall that a function  $E$  is Frechet differentiable at  $x \in \Omega$  if there exists a bounded linear functional, denoted by  $E'(x)$ , such that

$$\lim_{h \rightarrow 0} \frac{|E(x+h) - E(x) - \langle E'(x), h \rangle|}{\|h\|} = 0.$$

We start with discussing the connection between **Condition 1** and the modulus of uniform smoothness of  $E$  on  $\Omega$ .

### 2.1. Condition 1.

Given a convex function  $E : H \rightarrow \mathbb{R}$  and a set  $S \subset H$ , the modulus of smoothness of  $E$  on  $S$  is defined by

$$(2.4) \quad \rho(E, u) := \rho(E, u, S) := \frac{1}{2} \sup_{x \in S, \|y\|=1} \{E(x+uy) + E(x-uy) - 2E(x)\}, \quad u > 0,$$

and the modulus of uniform smoothness of  $E$  on  $S$  is defined by

$$(2.5) \quad \rho_1(E, u, S) := \sup_{x \in S, \|y\|=1, \lambda \in (0,1)} \left\{ \frac{(1-\lambda)E(x - \lambda uy) + \lambda E(x + (1-\lambda)uy) - E(x)}{\lambda(1-\lambda)} \right\}.$$

These two moduli of smoothness are equivalent (see [10], page 205):

**Lemma 2.1.** *Let  $E$  be a convex function defined on  $H$ , and let  $S \subset H$ , then*

$$(2.6) \quad 4\rho(E, \frac{u}{2}, S) \leq \rho_1(E, u, S) \leq 2\rho(E, u, S).$$

The next lemma shows the relation between the modulus of uniform smoothness and **Condition 1**.

**Lemma 2.2.** *Let  $E$  be a convex function defined on a Hilbert space  $H$  and  $E$  be Frechet differentiable on a set  $S \subset H$ . The following statements are equivalent for any  $q \in (1, 2]$  and  $M > 0$ .*

(i) *There exists  $\alpha > 0$ , such that for all  $x \in S, x' \in H, \|x - x'\| \leq M$ ,*

$$(2.7) \quad E(x') - E(x) - \langle E'(x), x' - x \rangle \leq \alpha \|x' - x\|^q.$$

(ii) *There exists  $\alpha_1 > 0$ , such that*

$$(2.8) \quad \rho(E, u, S) \leq \alpha_1 u^q, \quad 0 < u \leq M.$$

*The same result holds with  $\rho$  replaced by  $\rho_1$ .*

*Proof.* While this is a particular case of Corollary 3.5.7 from [10], for completeness of this paper, we provide a simple proof of this lemma. First, observe that because of Lemma 2.1, statement (ii) for  $\rho$  and  $\rho_1$  are equivalent, and so we can use them interchangeably. Assume that the first statement is true. For any  $x \in S, y \in H, \|y\| = 1$  and any  $0 < u \leq M$ , let  $x' := x + uy, x'' := x - uy$ . Then, we have  $\|x - x'\| = u \leq M, \|x'' - x\| = u \leq M$ . We apply (2.7) for the pairs  $(x', x)$  and  $(x'', x)$  to obtain

$$E(x + uy) - E(x) - u\langle E'(x), y \rangle \leq \alpha u^q, \quad E(x - uy) - E(x) + u\langle E'(x), y \rangle \leq \alpha u^q.$$

Therefore, we have

$$E(x + uy) + E(x - uy) - 2E(x) \leq 2\alpha u^q.$$

We take the supremum over  $x \in S, y \in H, \|y\| = 1$  and derive  $\rho(E, u, S) \leq \alpha u^q, 0 < u \leq M$ , which gives the lemma for  $\rho$ .

Conversely, suppose that (ii) holds for  $\rho_1$ . Then, for any  $\lambda \in (0, 1)$  and any  $x \in S, y \in H, \|y\| = 1, 0 < u \leq M$ ,

$$\frac{(1-\lambda)E(x - \lambda uy) + \lambda E(x + (1-\lambda)uy) - E(x)}{\lambda(1-\lambda)} \leq \alpha_1 u^q.$$

This is the same as saying

$$\frac{E(x - \lambda uy) - E(x)}{(1 - \lambda)\lambda} + \frac{E(x + (1 - \lambda)uy) - E(x - \lambda uy)}{1 - \lambda} \leq \alpha_1 u^q.$$

We let  $\lambda \rightarrow 0^+$  and use the continuity of  $E$  and the definition of Frechet derivative  $E'(x)$  with  $h = -\lambda uy$ , to obtain

$$\langle E'(x), -uy \rangle + E(x + uy) - E(x) \leq \alpha_1 u^q.$$

Now, for any  $x \in S$ ,  $x' \in H$ ,  $\|x' - x\| \leq M$ , we let  $u = \|x' - x\|$ ,  $y = \frac{x' - x}{\|x' - x\|}$ . The above inequality can be written as

$$E(x') - E(x) - \langle E'(x), x' - x \rangle \leq \alpha_1 \|x' - x\|^q,$$

which is (2.7) with  $\alpha = \alpha_1$ . □

## 2.2. Condition 2.

We first observe the following:

**Claim 1.** *If Condition 2 holds for a convex function  $E$  and a set  $\Omega$  that is convex and bounded, then Condition 2 holds for all  $x, x' \in \Omega$  with  $\beta$  replaced by  $\beta_0 > 0$ .*

*Proof.* Since  $\Omega$  is bounded, there is  $L > 0$ , such that  $\text{diam}(\Omega) \leq LM$ . Let  $x, x' \in \Omega$ . If  $\|x - x'\| \leq M$ , Condition 2 holds for the pair  $(x, x')$  provided  $\beta_0 \leq \beta$ . If  $\|x - x'\| > M$ , we chose a point  $x_1$ , such that

$$x_1 = \gamma x' + (1 - \gamma)x \in \Omega, \quad \gamma := \frac{M}{\|x - x'\|} \geq L^{-1}.$$

Clearly  $\|x - x_1\| = M$ , and therefore

$$E(x_1) - E(x) - \langle E'(x), x_1 - x \rangle \geq \beta \|x_1 - x\|^p.$$

Because of the convexity of  $E$ ,

$$E(x_1) - E(x) \leq \gamma[E(x') - E(x)].$$

A combination of the last two inequalities and the fact that  $x_1 - x = \gamma(x' - x)$  result in

$$E(x') - E(x) - \langle E'(x), x' - x \rangle \geq \beta \gamma^{p-1} \|x' - x\|^p \geq \beta L^{1-p} \|x' - x\|^p.$$

Therefore, the claim has been proven with  $\beta_0 = \min\{\beta, \beta L^{1-p}\}$ . □

Note that **Condition 2** is a generalization of the notion of strongly convex functions. Recall that a function  $E$  is called strongly convex on  $H$ , if there is a constant  $\beta > 0$ , called the convexity parameter of  $E$ , such that

$$E(x') - E(x) - \langle E'(x), x' - x \rangle \geq \beta \|x' - x\|^2, \quad x, x' \in H.$$

Next, we discuss the compatibility between the convexity of  $E$  and **Condition 2**.

**Lemma 2.3.** *Let  $E$  be a Frechet differentiable function on  $H$ .  $E$  is convex on  $H$  if and only if*

$$E(x') - E(x) - \langle E'(x), x' - x \rangle \geq 0, \quad \text{for all } x, x' \in H, \quad \|x - x'\| \leq M.$$

*Proof.* For convex functions on  $\mathbb{R}^n$ , a proof (without the restriction  $\|x' - x\| \leq M$ ) can be found in [2]. Simple modifications of this proof (which we do not give) result in a proof of the lemma.  $\square$

Finally, we present a concept which is dual to the modulus of uniform smoothness for convex functions, called the modulus of uniform convexity (see [1, 10]) and show how it is related to **Condition 2**. Given a convex function  $E : H \rightarrow \mathbb{R}$  and a set  $S \subset H$ , its modulus of uniform convexity on  $S$  is defined by

$$(2.9) \quad \delta_1(E, u, S) := \inf_{x \in S, \|y\|=1, \lambda \in (0,1)} \left\{ \frac{(1-\lambda)E(x - \lambda uy) + \lambda E(x + (1-\lambda)uy) - E(x)}{\lambda(1-\lambda)} \right\}.$$

We prove a lemma (see [10]) that shows the equivalence of **Condition 2** and certain behavior of the modulus of uniform convexity  $\delta_1$  of  $E$ .

**Lemma 2.4.** *Let  $E$  be a convex function defined on a Hilbert space  $H$  and  $E$  be Frechet differentiable on  $S \subset H$ . The following statements are equivalent for any  $p \in [2, \infty)$  and  $M > 0$ .*

(i) *There exists  $\beta > 0$ , such that for all  $x \in S$ ,  $x' \in H$ ,  $\|x - x'\| \leq M$ ,*

$$(2.10) \quad E(x') - E(x) - \langle E'(x), x' - x \rangle \geq \beta \|x' - x\|^p.$$

(ii) *There exists  $\beta_1 > 0$ , such that*

$$(2.11) \quad \delta_1(E, u, S) \geq \beta_1 u^p, \quad 0 < u \leq M.$$

*Proof.* Assume that the first statement is true. For any  $x \in S$ ,  $y \in H$ ,  $\|y\| = 1$ ,  $0 < u \leq M$  and  $\lambda \in (0, 1)$ , let  $x' := x - \lambda uy$ ,  $x'' := x + (1 - \lambda)uy$ . Then, we have  $\|x - x'\| = \lambda u \leq M$ ,  $\|x'' - x\| = (1 - \lambda)u \leq M$ . We apply (2.10) for  $x \in S$ ,  $x' \in H$  and  $x \in S$ ,  $x'' \in H$  to derive

$$E(x - \lambda uy) - E(x) + \lambda u \langle E'(x), y \rangle \geq \beta \lambda^p u^p,$$

$$E(x + (1 - \lambda)uy) - E(x) - (1 - \lambda)u \langle E'(x), y \rangle \geq \beta (1 - \lambda)^p u^p.$$

Multiplying the first inequality by  $(1 - \lambda)$ , the second one by  $\lambda$  and adding them yields

$$(1 - \lambda)E(x - \lambda uy) + \lambda E(x + (1 - \lambda)uy) - E(x) \geq \beta \lambda(1 - \lambda)(\lambda^{p-1} + (1 - \lambda)^{p-1})u^p.$$

Since  $\lambda^{p-1} + (1 - \lambda)^{p-1} \geq 2^{2-p}$  for  $\lambda \in (0, 1)$ , we have

$$\frac{(1 - \lambda)E(x - \lambda uy) + \lambda E(x + (1 - \lambda)uy) - E(x)}{\lambda(1 - \lambda)} \geq 2^{2-p} \beta u^p.$$

We take the infimum over  $x \in S$ ,  $y \in H$ ,  $\|y\| = 1$  and  $\lambda \in (0, 1)$  and obtain that  $\delta_1(E, u, S) \geq 2^{2-p}\beta u^p$ ,  $0 < u \leq M$ , which is (2.11) with  $\beta_1 = 2^{2-p}\beta$ .

Conversely, suppose that for some  $\beta > 0$  we have  $\delta_1(E, u, S) \geq \beta u^p$  for all  $0 < u \leq M$ . It follows from the definition of  $\delta_1$  that for any  $\lambda \in (0, 1)$ ,  $x \in S$ ,  $y \in H$ ,  $\|y\| = 1$  and  $0 < u \leq M$ ,

$$\frac{(1-\lambda)E(x - \lambda uy) + \lambda E(x + (1-\lambda)uy) - E(x)}{\lambda(1-\lambda)} \geq \beta_1 u^p.$$

This is the same as saying

$$\frac{E(x - \lambda uy) - E(x)}{\lambda} + \frac{E(x + (1-\lambda)uy) - E(x)}{1-\lambda} \geq \beta_1 u^p.$$

We let  $\lambda \rightarrow 0^+$  and by the continuity of  $E$  and the definition of Frechet derivative  $E'(x)$  for  $h = -\lambda uy$ , we obtain

$$\langle E'(x), -uy \rangle + E(x + uy) - E(x) \geq \beta_1 u^p.$$

Now, for any  $x \in S$ ,  $x' \in H$ ,  $\|x' - x\| \leq M$ , we let  $u = \|x' - x\|$ ,  $y = \frac{x' - x}{\|x' - x\|}$  and derive

$$E(x') - E(x) - \langle E'(x), x' - x \rangle \geq \beta_1 \|x' - x\|^p,$$

which is (2.10) with  $\beta = \beta_1$ .  $\square$

**2.3. The conditions on  $E$  and their connection to Compressed Sensing.** Let us summarize that as a result of Lemma 2.2 and Lemma 2.4, we have proven the following.

**Lemma 2.5.** *Let  $E$  be a convex function defined on a Hilbert space  $H$ . Let us denote by  $\Omega$  the set  $\Omega = \{x \in H : E(x) \leq E(0)\}$  and  $E$  be Frechet differentiable on  $\Omega$ . Let  $\delta_1(E, \cdot, \Omega)$  and  $\rho_1(E, \cdot, \Omega)$  be the modulus of uniform convexity and modulus of uniform smoothness of  $E$  on  $\Omega$ , respectively. The following two statements are equivalent*

(i)  *$E$  satisfies **Condition 1** and **Condition 2**.*

(ii) *There exist constants  $\alpha_1 > 0, \beta_1 > 0$ , such that*

$$\beta_1 u^p \leq \delta_1(E, u, \Omega) \leq \rho_1(E, u, \Omega) \leq \alpha_1 u^q, \quad u \in (0, M].$$

Let us next observe that (i) of the above lemma has a similar flavor to conditions that are imposed in compressed sensing. Indeed, conditions similar to **Condition 1** and **Condition 2** have been considered by Zhang in [12], where he solves a sparse optimization problem in  $\mathbb{R}^n$ , using greedy based strategies. He considers any convex function  $E$  on  $\mathbb{R}^n$  for which there are constants  $\alpha(s), \beta(s) > 0$  such that

$$(2.12) \quad \beta(s) \|x' - x\|_2^2 \leq E(x') - E(x) - \langle E'(x), x' - x \rangle \leq \alpha(s) \|x' - x\|_2^2,$$

holds whenever  $x, x' \in \mathbb{R}^n$  and  $x - x'$  has  $\leq s$  nonzero coordinates. Notice, that (2.12) is the same as our **Condition 1** and **Condition 2** except that it is only required to hold whenever  $x - x'$  is  $s$  sparse whereas in our case we require this to hold for all  $x, x'$  with  $\|x - x'\| \leq M$ . Zhang applied his results to the decoding problem in compressed sensing in which case  $E(x) = \|Ax - b\|_2^2$ , and  $A$  is a given  $k \times n$  matrix with  $k \ll n$ . For this choice of  $E$ , the Frechet derivative  $E'(x)$  can be computed explicitly as  $\langle E'(x), \cdot \rangle = 2\langle A^T(Ax - b), \cdot \rangle$ . Moreover, we have

$$\begin{aligned} E(x') - E(x) - \langle E'(x), x' - x \rangle &= \|Ax' - b\|_2^2 - \|Ax - b\|_2^2 - 2\langle A^T(Ax - b), x' - x \rangle \\ &= \|Ax' - Ax\|_2^2 = \|A(x - x')\|_2^2. \end{aligned}$$

If we denote by  $z = x' - x$ , condition (2.12) becomes

$$\beta(s)\|z\|_2^2 \leq \|Az\|_2^2 \leq \alpha(s)\|z\|_2^2,$$

for  $s$  sparse vectors  $z \in \mathbb{R}^n$ . This condition is known as the *Restricted Isometry Property* and was first introduced by Candes and Tao (see [3], [5]). For applications in compressed sensing one needs that  $\alpha(s), \beta(s)$  are sufficiently close to one.

### 3. GREEDY ALGORITHMS FOR OPTIMIZATION

In this section, we introduce the two algorithms for convex minimization in a Hilbert space  $H$  that we will analyze. As usual, we assume that  $\{\varphi_j\}_{j=1}^\infty$  is an orthonormal basis for  $H$ . We begin with the OMP(co) algorithm.

#### Orthogonal Matching Pursuit (OMP(co)):

- **Step 0:** Define  $x_0 := 0$ . If  $E'(x_0) = 0$ , stop the algorithm and define  $x_k := x_0$ ,  $k \geq 1$ .
- **Step  $m$ :** Assuming  $x_{m-1}$  has been defined and  $E'(x_{m-1}) \neq 0$ , Find

$$\varphi_{j_m} := \operatorname{argmax}\{|\langle E'(x_{m-1}), \varphi \rangle|, \varphi \in \mathcal{D}\},$$

and define

$$x_m := \operatorname{argmin}_{x \in \operatorname{span}\{\varphi_{j_1}, \varphi_{j_2}, \dots, \varphi_{j_m}\}} E(x).$$

If  $E'(x_m) = 0$ , stop the algorithm and define  $x_k := x_m$ ,  $k > m$ . Otherwise, go to **Step  $m + 1$** .

Note that if the algorithm stops at step  $m$ , then the output  $x_m$  of the algorithm is the minimizer  $\bar{x}$ , because of the following well-known lemma.

**Lemma 3.1.** *Let  $E$  be a Frechet differentiable convex function, defined on a convex domain  $\Omega$ . Then  $E$  has a global minimum at  $\bar{x} \in \Omega$  if and only if  $E'(\bar{x}) = 0$ .*



**Weak Chebyshev Greedy Algorithm (WCGA(co)):** The description of the WCGA(co) is the same as the OMP(co), with the only difference that a sequence  $\{t_k\}_{k=1}^\infty$ ,  $t_k \in (0, 1]$  is used to weaken the condition on the choice of  $\varphi_{j_m}$ . Namely,  $\varphi_{j_m}$  is now chosen to satisfy the inequality

$$|\langle E'(x_{m-1}), \varphi_{j_m} \rangle| \geq t_m \sup_{\varphi \in \mathcal{D}} \langle E'(x_{m-1}), \varphi \rangle.$$

When all  $t_k = 1$ ,  $k \geq 1$ , the WCGA(co) becomes the OMP(co).

Let us remark that neither of these two algorithms generates a unique sequence  $x_m$ ,  $m \geq 0$ . The analysis that follows applies to any sequence generated by the corresponding algorithm.

For comparison with the results we prove in this paper, we recall the result of Temlyakov. Let  $A_1(\mathcal{D})$  denote the closure (in  $H$ ) of the convex hull of  $\mathcal{D}$ . The following theorem was proved in [8] in a more general setting of Banach spaces and general symmetric dictionaries.

**Theorem 3.2** ([8] Theorem 2.2). *Let  $E$  be a uniformly smooth convex function defined on a Banach space  $X$  and let the set  $\Omega := \{x : E(x) \leq E(0)\}$  be bounded. Let the modulus of smoothness of  $E$  on  $\Omega$  satisfy  $\rho(E, u, \Omega) \leq \gamma u^q$ ,  $u > 0$ , where  $1 < q \leq 2$ . If for a given  $\epsilon > 0$ , there is an element  $\varphi^\epsilon \in \mathcal{D}$ , such that*

$$E(\varphi^\epsilon) \leq \inf_{x \in \Omega} E(x) + \epsilon, \quad \varphi^\epsilon / A(\epsilon) \in A_1(\mathcal{D}),$$

for some constant  $A(\epsilon) \geq 1$ , then, the output  $x_m^w$  of the WCGA satisfies the inequality

$$E(x_m^w) - \inf_{x \in \Omega} E(x) \leq \max \left\{ 2\epsilon, C_1 A(\epsilon)^q (C_2 + \sum_{k=1}^m t_k^{q/(q-1)})^{1-q} \right\},$$

with constants  $C_1 = C_1(q, \gamma)$  and  $C_2 = C_2(E, q, \gamma)$ .

#### 4. MAIN RESULTS

In this section, we present our main results and the auxiliary lemmas, needed for their proof. First, note that the set  $\Omega := \{x \in H : E(x) \leq E(0)\}$  is convex since it is the level set of a convex function. Also, all outputs  $\{x_k\}_{k=1}^\infty$  generated by the OMP(co) (or the WCGA(co)) are in  $\Omega$ , since the sequence  $\{E(x_k)\}_{k=1}^\infty$  is decreasing and  $E(x_1) \leq E(0)$ .

##### 4.1. Auxiliary lemmas.

Here, we begin with some lemmas that we use to derive our main results. The next lemma is well-known.

**Lemma 4.1.** *Let  $F$  be a Frechet differentiable function. Let  $V_k := \text{span}\{\varphi_{j_1}, \dots, \varphi_{j_k}\}$  and  $x_k := \text{argmin}\{F(x) : x \in V_k\}$ . Then, we have that  $\langle F'(x_k), \varphi \rangle = 0$  for every  $\varphi \in V_k$ .*

Our next lemma can be viewed as a generalization of Lemma 2.16 from [9].

**Lemma 4.2.** *Let  $\ell > 0$ ,  $r > 0$ ,  $B > 0$ ,  $\{a_m\}_{m=1}^\infty$  and  $\{r_m\}_{m=2}^\infty$  be sequences of non-negative numbers satisfying the inequalities*

$$a_1 \leq B, \quad a_{m+1} \leq a_m \left(1 - \frac{r_{m+1}}{r} a_m^\ell\right), \quad m = 1, 2, \dots$$

Then, we have

$$(4.13) \quad a_m \leq \max\{1, \ell^{-1/\ell}\} r^{1/\ell} (rB^{-\ell} + \sum_{k=2}^m r_k)^{-1/\ell}, \quad m = 2, 3, \dots$$

*Proof.* Let us first notice that from the recursive relation and the fact that all  $a_m$ 's are non-negative, we have

$$(4.14) \quad 0 \leq 1 - \frac{r_{m+1}}{r} a_m^\ell \leq 1, \quad m = 1, 2, \dots$$

We will show that for  $m = 2, 3, \dots$

$$(4.15) \quad a_m^\ell \leq \begin{cases} \frac{r}{(rB^{-\ell} + \sum_{k=2}^m r_k)}, & \text{if } \ell \geq 1, \\ \frac{r}{(rB^{-\ell} + \ell \sum_{k=2}^m r_k)}, & \text{if } 0 < \ell \leq 1, \end{cases}$$

from which the inequality (4.13) easily follows.

We prove (4.15) by induction.

**Case 1:**  $\ell \geq 1$ .

If  $a_2 = 0$ , then all  $a_m = 0$ ,  $m = 3, 4, \dots$ , and the lemma is true. Let us assume that  $a_2 > 0$ , and therefore  $a_1 > 0$ . It follows from the recursive relation and (4.14) that for  $\ell \geq 1$

$$a_2^{-\ell} \geq a_1^{-\ell} \left(1 - \frac{r_2}{r} a_1^\ell\right)^{-\ell} \geq a_1^{-\ell} \left(1 - \frac{r_2}{r} a_1^\ell\right)^{-1} \geq a_1^{-\ell} \left(1 + \frac{r_2}{r} a_1^\ell\right) = a_1^{-\ell} + \frac{r_2}{r} \geq B^{-\ell} + \frac{r_2}{r}.$$

This gives (4.15) for  $m = 2$ .

We now assume that (4.15) is true for  $m$  and prove it's validity for  $m + 1$ . As in the case  $m = 2$ , we may assume that  $a_{m+1} > 0$ . Because of the recursive relation, this also means that  $a_m > 0$  and using (4.14), we derive

$$(4.16) \quad a_{m+1}^{-\ell} \geq a_m^{-\ell} \left(1 - \frac{r_{m+1}}{r} a_m^\ell\right)^{-\ell} \geq a_m^{-\ell} \left(1 + \frac{r_{m+1}}{r} a_m^\ell\right) = a_m^{-\ell} + \frac{r_{m+1}}{r}.$$

Now, from the induction hypothesis we have that

$$a_m^{-\ell} \geq \frac{rB^{-\ell} + \sum_{k=2}^m r_k}{r},$$

which combined with (4.16) proves the lemma in the case  $\ell \geq 1$ .

**Case 2:**  $0 < \ell < 1$ .

Again, we need only consider the case when  $a_2 > 0$ . We will use the fact that for  $0 < \ell < 1$ , the function  $(1 - t)^\ell$  is concave. Therefore, we have

$$(4.17) \quad (1 - t)^\ell \leq 1 - \ell t, \quad 0 \leq t \leq 1.$$

We apply this inequality with  $t = \frac{r_2}{r} a_1^\ell \in [0, 1]$  and obtain

$$\begin{aligned} a_2^{-\ell} &\geq a_1^{-\ell} \left(1 - \frac{r_2}{r} a_1^\ell\right)^{-\ell} \geq a_1^{-\ell} \left(1 - \ell \frac{r_2}{r} a_1^\ell\right)^{-1} \geq a_1^{-\ell} \left(1 + \ell \frac{r_2}{r} a_1^\ell\right) \\ &= a_1^{-\ell} + \ell \frac{r_2}{r} \geq B^{-\ell} + \ell \frac{r_2}{r}, \end{aligned}$$

which gives (4.15) for  $m = 2$ . Next, we assume that (4.15) is true for  $m$  and prove it for  $m + 1$ . We can assume  $a_{m+1} > 0$  and therefore  $a_m > 0$ . From the recursive relation and (4.17) with  $t = \frac{r_{m+1}}{r} a_m^\ell \in [0, 1]$ , we have

$$\begin{aligned} a_{m+1}^{-\ell} &\geq a_m^{-\ell} \left(1 - \frac{r_{m+1}}{r} a_m^\ell\right)^{-\ell} \geq a_m^{-\ell} \left(1 - \ell \frac{r_{m+1}}{r} a_m^\ell\right)^{-1} \\ &\geq a_m^{-\ell} \left(1 + \ell \frac{r_{m+1}}{r} a_m^\ell\right) = a_m^{-\ell} + \ell \frac{r_{m+1}}{r}. \end{aligned}$$

This inequality, combined with the induction hypothesis gives that

$$a_{m+1}^{-\ell} \geq \frac{r B^{-\ell} + \ell \sum_{k=2}^{m+1} r_k}{r},$$

and the proof is complete.  $\square$

#### 4.2. Convergence rates for OMP(co).

In this section, we analyze the performance of the OMP(co) algorithm when applied to the minimization problem (1.1) with  $D = H$ . We assume that the dictionary  $\mathcal{D}$  is an orthonormal system  $\{\varphi_i\}_{i=1}^\infty$  and  $E$  takes on its global minimum  $\bar{x}$ . This means that this global minimum is assumed over  $\Omega := \{x : E(x) \leq E(0)\}$ . Let us denote by  $e_k$  the error of the algorithm at Step  $k$ , namely,

$$e_k := E(x_k) - E(\bar{x}).$$

The next lemma provides a recursive relation for the sequence  $\{e_k\}_{k=1}^\infty$ .

**Lemma 4.3.** *Let the objective function  $E$  satisfy **Conditions 0, 1, and 2**, and  $\mu$  be a constant such that  $\mu > \max\{1, M_0 \alpha^{-1} M^{1-q}\}$ . Let problem (1.1) have a solution  $\bar{x} = \sum_i c_i(\bar{x}) \varphi_i \in \Omega$  with support  $\bar{S} := \{i : c_i(\bar{x}) \neq 0\} < \infty$ , where  $\{\varphi_i\}$  is an orthonormal basis. Then, the error of the OMP(co) applied to  $E$  and  $\{\varphi_i\}$  satisfies the following recursive inequalities:*

$$(4.18) \quad e_1 \leq E(0) - E(\bar{x}),$$

and

$$(4.19) \quad e_k \leq e_{k-1} - \frac{(\mu - 1) \mu^{-q/(q-1)}}{r} e_{k-1}^{\frac{(p-1)q}{(q-1)p}}, \quad k \geq 2,$$

where the constant  $r$  is

$$r = |\bar{S}|^{\frac{q}{2(q-1)}} \alpha^{\frac{1}{q-1}} \left( p \beta_0^{1/p} (p-1)^{(1-p)/p} \right)^{-q/(q-1)}.$$

*Proof.* Clearly, we have  $e_1 = E(x_1) - E(\bar{x}) \leq E(0) - E(\bar{x})$  since

$$x_1 := \operatorname{argmin}_{x \in \operatorname{span}\{\varphi_{j_1}\}} E(x).$$

Next, we consider Step  $k$ ,  $k = 2, 3, \dots$  of the algorithm. Observe that if at Step  $(k-1)$  we have that  $\bar{S} \subseteq \{j_1, \dots, j_{k-1}\}$ , then  $x_{k-1} = \bar{x}$ ,  $E'(x_{k-1}) = 0$  and the OMP(co) would have stopped with output  $x_{k-1} = \bar{x}$ . If the algorithm has not stopped, then it generates the next output  $x_k$  and  $\varphi_{j_k}$ . Since  $x_k$  is the point of minimum of  $E$  over  $\operatorname{span}\{\varphi_{j_1}, \varphi_{j_2}, \dots, \varphi_{j_k}\}$ , we have for any  $|t| \leq M$ ,

$$(4.20) \quad E(x_k) \leq E(x_{k-1} + t\varphi_{j_k}) \leq E(x_{k-1}) + t \langle E'(x_{k-1}), \varphi_{j_k} \rangle + \alpha |t|^q,$$

where the last inequality invoked **Condition 1**. We take

$$t = -(\alpha\mu)^{-\frac{1}{q-1}} \operatorname{sign}(\langle E'(x_{k-1}), \varphi_{j_k} \rangle) |\langle E'(x_{k-1}), \varphi_{j_k} \rangle|^{\frac{1}{q-1}}.$$

Because of the definition of  $\mu$  in the statement of the theorem, we have  $|t| \leq M$ . Therefore, we have

$$(4.21) \quad E(x_k) \leq E(x_{k-1}) - \frac{\mu - 1}{\mu} (\alpha\mu)^{-\frac{1}{q-1}} |\langle E'(x_{k-1}), \varphi_{j_k} \rangle|^{q/(q-1)}.$$

Now, we will find a lower bound for  $|\langle E'(x_{k-1}), \varphi_{j_k} \rangle|$ . First, note that from **Condition 2** and Claim 1 applied to  $x' = \bar{x}$  and  $x = x_{k-1}$  (both are in  $\Omega$ ), we obtain

$$(4.22) \quad \langle E'(x_{k-1}), x_{k-1} - \bar{x} \rangle \geq E(x_{k-1}) - E(\bar{x}) + \beta_0 \|\bar{x} - x_{k-1}\|^p.$$

Let us recall the weighted arithmetic mean -geometric mean inequality

$$\frac{p_1}{p_1 + p_2} a + \frac{p_2}{p_1 + p_2} b \geq a^{\frac{p_1}{p_1 + p_2}} b^{\frac{p_2}{p_1 + p_2}}, \quad \text{where } a, b \geq 0, \quad p_1, p_2 > 0,$$

and apply it for  $p_1 = p - 1$ ,  $p_2 = 1$ ,  $a = \frac{E(x_{k-1}) - E(\bar{x})}{p-1}$ ,  $b = \beta_0 \|\bar{x} - x_{k-1}\|^p$ . We have

$$E(x_{k-1}) - E(\bar{x}) + \beta_0 \|\bar{x} - x_{k-1}\|^p = p \left( \frac{(p-1)}{p} \frac{E(x_{k-1}) - E(\bar{x})}{p-1} + \frac{1}{p} \beta_0 \|\bar{x} - x_{k-1}\|^p \right),$$

and therefore

$$E(x_{k-1}) - E(\bar{x}) + \beta_0 \|\bar{x} - x_{k-1}\|^p \geq C \|\bar{x} - x_{k-1}\| (E(x_{k-1}) - E(\bar{x}))^{(p-1)/p},$$

with  $C = p \beta_0^{1/p} (p-1)^{(1-p)/p}$ . We combine this inequality with (4.22) to obtain

$$(4.23) \quad \langle E'(x_{k-1}), x_{k-1} - \bar{x} \rangle \geq C \|\bar{x} - x_{k-1}\| (E(x_{k-1}) - E(\bar{x}))^{(p-1)/p}.$$

From the definition of  $x_{k-1}$  and Lemma 4.1, it follows that

$$\langle E'(x_{k-1}), \varphi_i \rangle = 0, \quad i = j_1, \dots, j_{k-1}.$$

Therefore, if we write

$$x_{k-1} - \bar{x} = \sum_i c_i(x_{k-1} - \bar{x})\varphi_i,$$

since the support of  $x_{k-1}$  is  $\{j_1, \dots, j_{k-1}\}$ , we obtain

$$\begin{aligned} \langle E'(x_{k-1}), x_{k-1} - \bar{x} \rangle &= \sum_{i \in \bar{S} \setminus \{j_1, \dots, j_{k-1}\}} c_i(x_{k-1} - \bar{x}) \langle E'(x_{k-1}), \varphi_i \rangle, \\ (4.24) \quad &\leq \sum_{i \in \bar{S} \setminus \{j_1, \dots, j_{k-1}\}} |c_i(x_{k-1} - \bar{x})| |\langle E'(x_{k-1}), \varphi_{j_k} \rangle| \\ &\leq |\langle E'(x_{k-1}), \varphi_{j_k} \rangle| |\bar{S}|^{1/2} \|x_{k-1} - \bar{x}\|. \end{aligned}$$

We combine this inequality with (4.23) and derive that

$$|\langle E'(x_{k-1}), \varphi_{j_k} \rangle| \|\bar{x} - x_{k-1}\| |\bar{S}|^{1/2} \geq C \|\bar{x} - x_{k-1}\| (E(x_{k-1}) - E(\bar{x}))^{(p-1)/p}.$$

Therefore we have the desired lower bound

$$|\langle E'(x_{k-1}), \varphi_{j_k} \rangle| \geq C |\bar{S}|^{-1/2} (E(x_{k-1}) - E(\bar{x}))^{(p-1)/p}.$$

The latter result and (4.21) gives the estimate

$$E(x_k) \leq E(x_{k-1}) - \frac{(\mu - 1)C^{q/(q-1)}}{\mu^{q/(q-1)}\alpha^{1/(q-1)}|\bar{S}|^{\frac{q}{2(q-1)}}} (E(x_{k-1}) - E(\bar{x}))^{\frac{(p-1)q}{(q-1)p}}.$$

Subtracting  $E(\bar{x})$  from both sides of this inequality results in (4.19) and the proof is completed.  $\square$

We next remark that we can take a specific value for  $\mu$  in the last lemma.

**Remark 4.4.** *Let the objective function  $E$  satisfy **Conditions 0, 1, and 2**. Let problem (1.1) have a solution  $\bar{x} = \sum_i c_i(\bar{x})\varphi_i \in \Omega$  with support  $\bar{S} := \{i : c_i(\bar{x}) \neq 0\} < \infty$ , where  $\{\varphi_i\}$  is an orthonormal basis. Then, the error of the OMP(co) applied to  $E$  and  $\{\varphi_i\}$  satisfies the following recursive inequalities:*

$$e_1 \leq E(0) - E(\bar{x}),$$

and

$$(4.25) \quad e_k \leq e_{k-1} - \frac{C_3}{r} e_{k-1}^{\frac{(p-1)q}{(q-1)p}} = e_{k-1} [1 - \frac{C_3}{r} e_{k-1}^{\frac{p-q}{(q-1)p}}], \quad k \geq 2,$$

where  $r$  is the constant from Lemma 4.3 and  $C_3 = C_3(M_0, M, \alpha, q)$  is

$$(4.26) \quad C_3 = \begin{cases} (q-1)q^{-q/(q-1)}, & \text{if } M_0 M^{1-q} \alpha^{-1} < q, \\ (M_0 M^{1-q} \alpha^{-1} - 1) M_0^{-q/(q-1)} M^{-q} \alpha^{q/(q-1)}, & \text{if } M_0 M^{1-q} \alpha^{-1} \geq q. \end{cases}$$

*Proof.* The estimate follows from Lemma 4.3 and the fact that the function

$$g(\mu) = (\mu - 1)\mu^{-q/(q-1)}$$

is increasing on  $(1, q)$  and decreasing on  $(q, \infty)$  with global maximum at  $\mu = q$ .  $\square$

The next theorem is our main result about the OMP(co) algorithm.

**Theorem 4.5.** *Let the objective function  $E$  satisfy **Conditions 0**, **1**, and **2**. Let problem (1.1) with  $D = \Omega := \{x : E(x) \leq E(0)\}$  have a solution  $\bar{x} = \sum_i c_i(\bar{x})\varphi_i \in \Omega$  with support  $\bar{S} := \{i : c_i(\bar{x}) \neq 0\} < \infty$ , where  $\{\varphi_i\}$  is an orthonormal basis for  $H$ . Then, at Step  $k$ , the OMP(co) applied to  $E$  and  $\{\varphi_i\}$  outputs  $x_k$ , where either  $x_k = \bar{x}$ , in which case  $e_k = 0$ , or:*

(i) *When  $p \neq q$ , for  $k = 2, 3, \dots$ ,*

$$e_k \leq C|\bar{S}|^{\frac{pq}{2(p-q)}} k^{-\frac{p(q-1)}{p-q}},$$

$$\|x_k - \bar{x}\| \leq C'|\bar{S}|^{\frac{q}{2(p-q)}} k^{-\frac{q-1}{p-q}}$$

where  $C$  and  $C'$  depend only on  $p, q, \alpha, \beta, E$ .

(ii) *When  $p = q = 2$ , we have the exponential decay*

$$e_k \leq C_2 \gamma^{k-1},$$

$$\|x_k - \bar{x}\| \leq C_2^{\frac{1}{2}} \beta_0^{-\frac{1}{2}} \gamma^{(k-1)/2}, \quad k = 2, 3, \dots,$$

where  $\gamma := 1 - \frac{\tilde{C}_3}{|\bar{S}|}$  is in  $(0, 1)$ ,  $C_2 = E(0) - E(\bar{x})$ , and  $\tilde{C}_3$  is a constant that depends on  $\alpha, \beta$ , and  $E$ .

*Proof.* In the case  $p \neq q$ , we define the sequence of non-negative numbers

$$r_k = C_3, \quad a_k = E(x_k) - E(\bar{x}), \quad k = 1, 2, \dots,$$

and the numbers

$$r = |\bar{S}|^{\frac{q}{2(q-1)}} \alpha^{\frac{1}{q-1}} (p\beta^{1/p}(p-1)^{(1-p)/p})^{-q/(q-1)} > 0,$$

$$\ell = \frac{p-q}{p(q-1)} > 0, \quad B = E(0) - E(\bar{x}) > 0.$$

It follows from Remark 4.4 that the above defined sequences satisfy the conditions of Lemma 4.2, and therefore we have

$$(4.27) \quad e_k = E(x_k) - E(\bar{x}) \leq C_0 \left( \frac{|\bar{S}|^{\frac{q}{2(q-1)}}}{C_1 |\bar{S}|^{\frac{q}{2(q-1)}} + C_3(k-1)} \right)^{\frac{p(q-1)}{p-q}},$$

where

$$C_0 = C_0(p, q, \alpha, \beta) = \alpha^{\frac{p}{p-q}} (p\beta^{1/p}(p-1)^{(1-p)/p})^{-\frac{pq}{p-q}} \cdot \max \left\{ 1, \left( \frac{p(q-1)}{p-q} \right)^{\frac{p(q-1)}{p-q}} \right\},$$

and

$$C_1 = C_1(p, q, \alpha, \beta, E) = \alpha^{\frac{1}{q-1}} (p\beta^{1/p}(p-1)^{(1-p)/p})^{-q/(q-1)} (E(0) - E(\bar{x}))^{\frac{q-p}{p(q-1)}}.$$

One easily derives the estimate for  $e_k$  in (i) from (4.27). The estimate for  $\|x_k - \bar{x}\|$  in (i) now follows from **Condition 2** with  $x' = x_k$ ,  $x = \bar{x}$  and Lemma 3.1.

In the case  $p = q = 2$ , as before  $E(x_1) - E(\bar{x}) \leq E(0) - E(\bar{x})$ , and Lemma 4.3 and Remark 4.4 give that

$$E(x_k) - E(\bar{x}) \leq \left(1 - \frac{\tilde{C}_3}{|\bar{S}|}\right) (E(x_{k-1}) - E(\bar{x})), \quad k = 2, 3, \dots,$$

where

$$(4.28) \quad \tilde{C}_3 = \begin{cases} \frac{\beta_0}{\alpha}, & \text{if } M_0 M^{-1} \alpha^{-1} < 2, \\ 4\beta_0(M_0 M^{-1} \alpha^{-1} - 1)M_0^{-2}M^{-2}\alpha, & \text{if } M_0 M^{-1} \alpha^{-1} \geq 2. \end{cases}$$

It follows that

$$E(x_k) - E(\bar{x}) \leq (E(0) - E(\bar{x})) \left(1 - \frac{\tilde{C}_3}{|\bar{S}|}\right)^{k-1}, \quad k = 2, 3, \dots$$

As in the previous case, we use **Condition 2** with  $x' = x_k$ ,  $x = \bar{x}$  and Lemma 3.1 to derive the estimate for  $\|x_k - \bar{x}\|$ .  $\square$

**4.3. Main results for WCGA(co).** The convergence analysis of the WCGA(co) is almost the same as that for the OMP(co). We omit the details here and just state the error estimates for

$$e_k^w := E(x_k^w) - E(\bar{x}),$$

pointing out the main differences in the proof.

**Theorem 4.6.** *Let the objective function  $E$  satisfy **Conditions 0, 1, and 2**. Let problem (1.1) with  $D = \Omega = \{x : E(x) \leq E(0)\}$  have a solution  $\bar{x} = \sum_i c_i(\bar{x})\varphi_i \in \Omega$  with support  $\bar{S} := \{i : c_i(\bar{x}) \neq 0\} < \infty$ , where  $\{\varphi_i\}$  is an orthonormal basis. Then, at Step  $k$ , the WCGA applied to  $E$  and  $\{\varphi_i\}$  outputs  $x_k^w$ , where either  $x_k^w = \bar{x}$ , in which case  $e_k^w = 0$ , or:*

(i) *When  $p \neq q$ , for each  $k = 2, 3, \dots$ , we have*

$$e_k^w \leq \tilde{C} |\bar{S}|^{\frac{pq}{2(p-q)}} \left( \sum_{j=2}^k t_j^{\frac{q}{q-1}} \right)^{\frac{p(q-1)}{p-q}}$$

$$\|x_k^w - \bar{x}\| \leq \tilde{C}' |\bar{S}|^{\frac{q}{2(p-q)}} \left( \sum_{j=2}^k t_j^{\frac{q}{q-1}} \right)^{\frac{(q-1)}{p-q}}$$

where  $\tilde{C}$  and  $\tilde{C}'$  depend only on  $p, q, \alpha, \beta, E$ .

(ii) When  $p = q = 2$ , we have

$$e_k^w \leq C_2 \prod_{j=2}^k \left( 1 - \frac{\tilde{C}_3}{|\bar{S}|} t_j^2 \right),$$

$$\|x_k^w - \bar{x}\| \leq C_2^{\frac{1}{2}} \beta^{-\frac{1}{2}} \prod_{j=2}^k \left( 1 - \frac{\tilde{C}_3}{|\bar{S}|} t_j^2 \right)^{1/2},$$

with  $C_2 = E(0) - E(\bar{x})$  and  $\tilde{C}_3$  depends on  $\alpha, \beta$ , and  $E$ .

*Proof.* The proof follows the lines of that of Theorem 4.5 and the corresponding lemmas. The difference is that instead of estimate (4.24), we have

$$\begin{aligned} \langle E'(x_{k-1}^w), x_{k-1}^w - \bar{x} \rangle &= \sum_{i \in \bar{S} \setminus j_1, \dots, j_{k-1}} c_i(x_{k-1}^w - \bar{x}) \langle E'(x_{k-1}^w), \varphi_i \rangle \\ &\leq \sum_{i \in \bar{S} \setminus j_1, \dots, j_{k-1}} |c_i(x_{k-1}^w - \bar{x})| |\langle E'(x_{k-1}^w), \varphi_i \rangle| \\ &\leq t_k^{-1} |\langle E'(x_{k-1}^w), \varphi_{j_k} \rangle| \sum_{i \in \bar{S}} |c_i(x_{k-1}^w - \bar{x})| \\ (4.29) \quad &\leq t_k^{-1} |\langle E'(x_{k-1}^w), \varphi_{j_k} \rangle| \|\bar{x} - x_{k-1}^w\| |\bar{S}|^{1/2}, \end{aligned}$$

and that we use Lemma 4.2 with  $r_k = C_3 t_k^{\frac{q}{q-1}}$ ,  $k = 2, 3, \dots$  □

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